LEVEL L

L 1-10 : Logarithmic Functions

$a^n = N \quad \Rightarrow \quad \log_a N = n.$

$2^3 = 8 \quad \Rightarrow \quad \log_2 8 = 3$; $3^{-2} = \frac{1}{9} \quad \Rightarrow \quad \log_3 \frac{1}{9} = -2$

When solving equations, first rewrite as exponents

$\log_x 9 = 2$	$\log_{\sqrt{3}} 9 = x$
[Sol] $x^2 = 9 = 3^2$	[Sol] $(\sqrt{3})^x = 9$
Since $x > 0$,	$3^{\frac{x}{2}} = 3^2$
$x = 3$	$x = 4$

Important properties of logarithms:

Properties of Logarithms

When $a > 0,\ a \neq 1,\ M > 0,\ N > 0$,

1. $\log_a 1 = 0,\ \ \log_a a = 1,\ \ \log_a a^m = m$

2. $\log_a MN = \log_a M + \log_a N$

3. $\log_a \frac{M}{N} = \log_a M - \log_a N \ \left(\log_a \frac{1}{N} = -\log_a N\right)$

4. $\log_a M^n = n \log_a M \ \left(\log_a \sqrt[n]{M} = \frac{1}{n}\log_a M\right)$

Application of Property (1) and (2)

$$\begin{aligned}
&\log_3 54 + \log_3 6 - 2\log_3 2 \\
&= \log_3(2\cdot 3^3) + \log_3(2\cdot 3) - 2\log_3 2 \\
&= \log_3 2 + 3 + \log_3 2 + 1 - 2\log_3 2 \\
&= 4
\end{aligned}$$

Application of Property (2) and (4)

$$\begin{aligned}
&\log_{10} 25 + \log_{10} 2 - \frac{1}{2}\log_{10} 25 \\
&= 2\log_{10} 5 + \log_{10} 2 - \log_{10} 5 \\
&= \log_{10} 5 + \log_{10} 2 \\
&= 1
\end{aligned}$$

Important formula

Logarithmic Base Conversion Formula

$$\log_a M = \frac{\log_b M}{\log_b a}$$

(where $a > 0,\ b > 0,\ M > 0$ and $a \neq 1,\ b \neq 1$)

$$\begin{aligned}
&(\log_2 3 + \log_4 9)(\log_3 2 + \log_9 4) \\
&= \left(\log_2 3 + \frac{\log_2 9}{\log_2 4}\right)\left(\frac{\log_2 2}{\log_2 3} + \frac{\log_2 4}{\log_2 9}\right) \\
&= \left(\log_2 3 + \frac{2\log_2 3}{2}\right)\left(\frac{1}{\log_2 3} + \frac{2}{2\log_2 3}\right) \\
&= 2\log_2 3 \cdot \frac{2}{\log_2 3} \\
&= \mathbf{4}
\end{aligned}$$

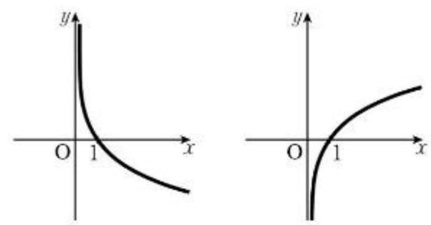
Base conversion formula & Property (4)

L 11-20 : Graphs of Logarithmic Functions

A logarithmic function is the **inverse function** of exponential function. The domain is $x > 0$

The graph of the logarithmic function $y = \log_a x$ passes through point $(1,\ 0)$, and its asymptote is the y-axis.

When $0 < a < 1$ When $a > 1$



The graph $y = \log_a(x - p) + q$ (**where $a > 0$ and $a \neq 1$**) is a translation of $= \log_a x$, p units along the x-axis and q units along the y-axis.

Since a logarithmic function is monotonic,

Given the logarithmic functions $y = \log_a p$ and $y = \log_a q$:

When $a > 1$, then $p < q \iff \log_a p < \log_a q$.

When $0 < a < 1$, then $p < q \iff \log_a p > \log_a q$.

Example:

Order the following logarithms.

$\log_9 12,\quad 2\log_3 2$

[Sol] $\log_9 12 = \frac{\log_3 12}{\log_3 9} = \frac{1}{2}\log_3 12 = \log_3 12^{\frac{1}{2}}$,

$2\log_3 2 = \log_3 4$

The base is > 1, and $12^{\frac{1}{2}} < 4$.

Therefore, $\log_3 12^{\frac{1}{2}} < \log_3 4$

Thus, $\mathbf{\log_9 12 < 2\log_3 2}$

L 21-30 : Logarithmic Equations & Inequalities

Use the properties of logarithms and the domain of logarithmic function when solving equations

$\log_{10}x+\log_{10}(x+3)=1$

$\log_{10}x(x+3)=1$

$x(x+3)=10$

$x^2+3x-10=0$

$x=-5,\ 2$

Since antilogarithms are >0,

$x>0,\quad x+3>0$

So we must have $x>0$.

The solution $x=-5$ does not satisfy this condition.

Therefore, $\boldsymbol{x=2}$

$\log_2(x+1)=\log_2(2-x)+1$

$\log_2(x+1)-\log_2(2-x)=1$

$\log_2\dfrac{x+1}{2-x}=1$

$\dfrac{x+1}{2-x}=2$

$x=1$

Since antilogarithms are >0,

$x+1>0,\quad 2-x>0$

So we must have $-1<x<2$.

The solution $x=1$ satisfies this condition.

Therefore, $\boldsymbol{x=1}$

Sometimes, we can solve logarithmic equations by recognising a quadratic equation in $\log_a x$

$(1+\log_2 x)\cdot\log_2 x=2$

Let $\log_2 x=X$

$(1+X)X=2$

$X^2+X-2=0$

$X=-2,\ 1$

When $X=-2$, $\log_2 x=-2$, $x=\dfrac{1}{4}$

When $X=1$, $\log_2 x=1$, $x=2$

$2\log_2 x-3\log_x 2+5=0$

Let $\log_2 x=X$

Since $x\neq 1$, then $\log_2 x=X\neq 0$

Therefore,

$2X-\dfrac{3}{X}+5=0$

$2X^2+5X-3=0$

$X=\dfrac{1}{2},-3$

When $X=\dfrac{1}{2}$, $\log_2 x=\dfrac{1}{2}$, $\boldsymbol{x=\sqrt{2}}$

When $X=-3$, $\log_2 x=-3$, $\boldsymbol{x=\dfrac{1}{8}}$

When solving inequalities, we switch the inequality sign if the base is < 1, leave the inequality sign unchanged if the base is > 1.

$\log_3(x-2)+\log_3(x-4)<1$

$\log_3(x-2)(x-4)<1$

Since the base is >1,

$(x-2)(x-4)<3$

$x^2-6x+5<0$

$(x-1)(x-5)<0$

$1<x<5\quad\cdots①$

Since antilogarithms are >0,

$x-2>0,\quad x-4>0$

So we must also have $x>4\quad\cdots②$

From ① and ②,

$\boldsymbol{4<x<5}$

$2\log_{\frac{1}{2}}(x-2)>\log_{\frac{1}{2}}(x+4)$

$\log_{\frac{1}{2}}(x-2)^2>\log_{\frac{1}{2}}(x+4)$

Since the base is <1,

$(x-2)^2<x+4$

$x^2-5x<0$

$x(x-5)<0$

$0<x<5\quad\cdots①$

Since antilogarithms are >0,

$x-2>0,\quad x+4>0$

So we must also have $x>2\quad\cdots②$

From ① and ②,

$\boldsymbol{2<x<5}$

L 31-40 : Modulus Functions

$$|f(x)|=\begin{cases} f(x) & for\ x\ with\ f(x)\geq 0 \\ -f(x) & for\ x\ with\ f(x)<0\end{cases}$$

Examples:

$$|(x-1)(x-3)|=\begin{cases} (x-1)(x-3) & (\text{when } x\leqslant 1 \text{ or } x\geqslant 3) \\ -(x-1)(x-3) & (\text{when } 1<x<3)\end{cases}$$

$$|x^2-9|=\begin{cases} x^2-9 & (\text{when } x\leqslant -3 \text{ or } x\geqslant 3) \\ -(x^2-9) & (\text{when } -3<x<3)\end{cases}$$

The graphs of modulus functions always lie above or on the x-axis.

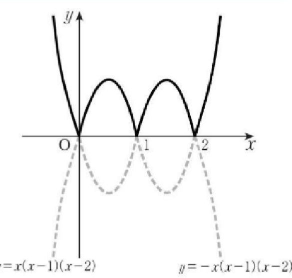
$y=|x(x-1)(x-2)|$

[Sol] When $x(x-1)(x-2)\geqslant 0$,

i.e $\boxed{\text{when } 0\leqslant x\leqslant 1 \text{ or } x\geqslant 2,\ y=x(x-1)(x-2)}$

When $x(x-1)(x-2)<0$,

i.e $\boxed{\text{when } x<0 \text{ or } 1<x<2,\ y=-x(x-1)(x-2)}$



L 41-50 : Limits & Derivatives

$\lim\limits_{x\to a} f(x)$ is the limit value of $f(x)$ as x approaches a, but not equal to a.

$$\lim_{x\to 1}\frac{x^3-x^2+x-1}{x^2+x-2}=\lim_{x\to 1}\frac{(x-1)(x^2+1)}{(x+2)(x-1)}=\lim_{x\to 1}\frac{x^2+1}{x+2}=\boldsymbol{\frac{2}{3}}$$

$$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=\boldsymbol{3x^2}$$

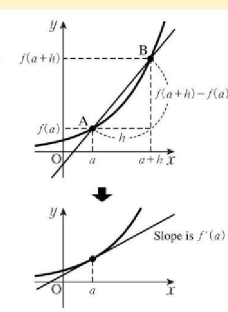
By taking the limit of the average rate of change over an interval, we can obtain the instantaneous rate of change at a specific point.

Given $y=f(x)$, the average rate of change as x varies from a to $a+h$ can be expressed as $\dfrac{f(a+h)-f(a)}{h}$. In this case,

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

($f'(a)$ is read as "f prime of a".)

$f'(a)$ is the ***differential coefficient*** at $x=a$.



The derivative of $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Rules of Differentiation:

When n is a natural number, $(x^n)' = nx^{n-1}$.

Rules of Differentiation

1. If $y = k$, then $y' = 0$
2. If $y = kf(x)$, then $y' = kf'(x)$
3. If $y = f(x) + g(x)$, then $y' = f'(x) + g'(x)$
4. If $y = f(x) - g(x)$, then $y' = f'(x) - g'(x)$
5. If $y = f(x)g(x)$, then $y' = f'(x)g(x) + f(x)g'(x)$

Note: Rule 5 is known as the **product rule** (revisit in level N and O)

(1) $y = 2x^4 - 3x^3 + 2x^2$

$\boldsymbol{y' = 8x^3 - 9x^2 + 4x}$

(2) $y = 5 - 3x + 4x^2 - 2x^5$

$\boldsymbol{y' = -3 + 8x - 10x^4}$

(3) $y = (2x^2 + 1)(3x^2 - 1)$

$\boldsymbol{y'} = 4x(3x^2 - 1) + (2x^2 + 1) \cdot 6x$

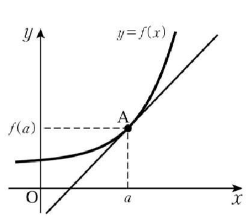
$= 12x^3 - 4x + 12x^3 + 6x$

$\boldsymbol{= 24x^3 + 2x}$

L 51-60 : Tangents

The gradient of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$.
The equation of the tangent at point $(a, f(a))$ is as follows:

$$y - f(a) = f'(a)(x - a)$$



1) Identify the point of tangency (sometimes can be an unknown).

2) Find the derivative of the function evaluated at the x-coordinate of the point of tangency.

3) Express the equation of the tangent in point-slope form.

$y = x^3 - 2x - 2 \quad (x = 1)$

Let $f(x) = x^3 - 2x - 2$

$f'(x) = 3x^2 - 2$

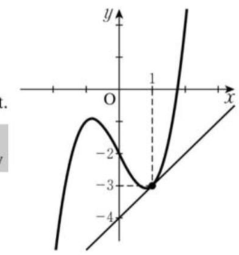
$f'(1) = 1$ The gradient of the tangent.

Also, $f(1) = -3$ The y-coordinate of the point of tangency

Therefore, the tangent is:

$y + 3 = 1 \cdot (x - 1)$

$y = x - 4$



Given the curve $y = x^3 - 2x + 1$, find the points of contact where the tangents have gradient 1. Find the tangents passing through the points of contact.

[Sol] Let $f(x) = x^3 - 2x + 1 \Rightarrow f'(x) = 3x^2 - 2$

From $3x^2 - 2 = 1$,

$x = \pm 1$

$f(1) = 0, \quad f(-1) = 2$

Therefore, the points of contact are: **(1, 0), (−1, 2)**

When the point of contact is $(1, 0)$, the tangent is:

$y - 0 = 1 \cdot (x - 1)$

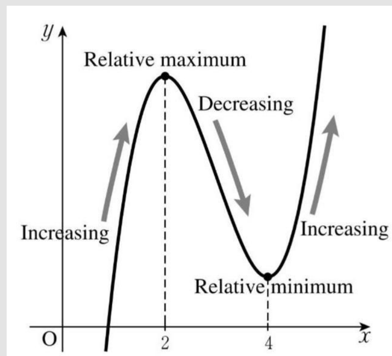
$\boldsymbol{y = x - 1}$

When the point of contact is $(-1, 2)$, the tangent is:

$y - 2 = 1 \cdot (x + 1)$

$\boldsymbol{y = x + 3}$

L 61-80 : Relative Maxima & Minima



To identify the relative maximum/minimum of a function, we use a **variation table** (with x, y, y')

Given the function $y = x^3 - 3x + 1$

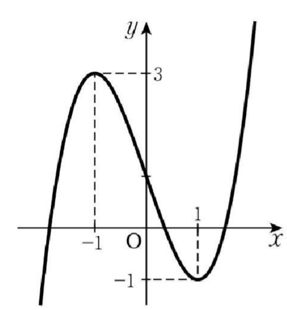
From $y' = 3x^2 - 3 = 3(x+1)(x-1) = 0$

$x = -1, 1$

The variation table

x	$\cdots$	-1	$\cdots$	**1**	$\cdots$
y'	+	0	−	**0**	+
y	↗	3	↘	**−1**	↗
		(relative maximum)		(relative minimum)	

relative maximum value is 3, at $x = -1$,
relative minimum value is -1, at $x = 1$.



1) Evaluate y', equate to zero and find the critical points. (i.e. values of x such that $y' = 0$). Include the critical points in the variation table.

2) Determine the sign of y' in each interval. Remember than the function is increasing (upward sloping) if $y' > 0$ and decreasing (downward sloping) if $y' < 0$.

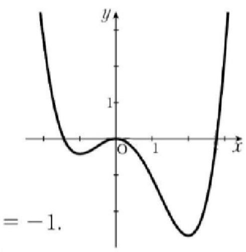
3) Indicate the relative/local maxima and minima.

$y = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2$

From $y' = x^3 - x^2 - 2x = x(x-2)(x+1) = 0$,

$x = -1,\ 0,\ 2$

x	$\cdots$	-1	$\cdots$	0	$\cdots$	2	$\cdots$
y'	$-$	0	$+$	0	$-$	0	$+$
y	$\searrow$	$-\frac{5}{12}$	$\nearrow$	0	$\searrow$	$-\frac{8}{3}$	$\nearrow$



Therefore, we have:

A relative minimum value of $-\frac{5}{12}$, at $x = -1$.

A relative maximum value of 0, at $x = 0$.

A relative minimum value of $-\frac{8}{3}$, at $x = 2$.

<u>Tips:</u> Recall the shapes of the graph of a polynomial function (K101–110) would be useful.

When we have arbitrary constant a in the function, always consider different ranges of values of a.

Given the function $f(x) = ax^3 + 2ax^2 + ax + 1 \quad (a \neq 0)$

From $f'(x) = 3ax^2 + 4ax + a = a(3x+1)(x+1) = 0$,

$x = -1,\ -\frac{1}{3}$

When $a > 0$

When $a < 0$

x	$\cdots$	-1	$\cdots$	$-\frac{1}{3}$	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	relative maximum	$\searrow$	relative minimum	$\nearrow$

x	$\cdots$	-1	$\cdots$	$-\frac{1}{3}$	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	relative minimum	$\nearrow$	relative maximum	$\searrow$

$f(x) = -2x^3 + 3(a+2)x^2 - 12ax + 4a^2$

From $f'(x) = -6x^2 + 6(a+2)x - 12a = -6[x^2 - (a+2)x + 2a]$

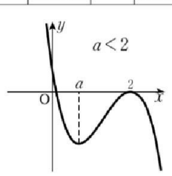
$= -6(x-a)(x-2) = 0,\ x = a,\ 2$

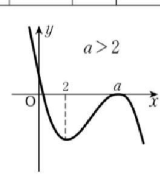
When $a < 2$,

x	$\cdots$	a	$\cdots$	2	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	relative minimum	$\nearrow$	relative maximum	$\searrow$

When $a > 2$,

x	$\cdots$	2	$\cdots$	a	$\cdots$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	relative minimum	$\nearrow$	relative maximum	$\searrow$

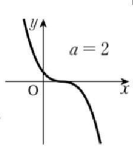




When $a = 2$,

from $f'(x) = -6(x-2)^2 \leq 0$,

there are no relative extreme values.



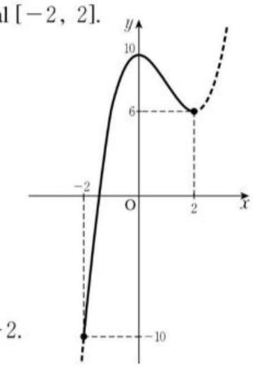
L 81-100 : (Absolute) Maxima & Minima

On a given domain/interval, the maximum point of a function is the highest point attainable, similarly the minimum point is the lowest point attainable.

Find the maximum and minimum values of the function $y = x^3 - 3x^2 + 10$ on the closed interval $[-2,\ 2]$.

$y' = 3x^2 - 6x = 3x(x-2)$

x	$\boxed{-2}$	$\cdots$	0	$\cdots$	$\boxed{2}$
y'	$+$	$+$	0	$-$	0
y	-10	$\nearrow$	10	$\searrow$	6



Therefore:

The maximum value is 10, at $x = 0$.

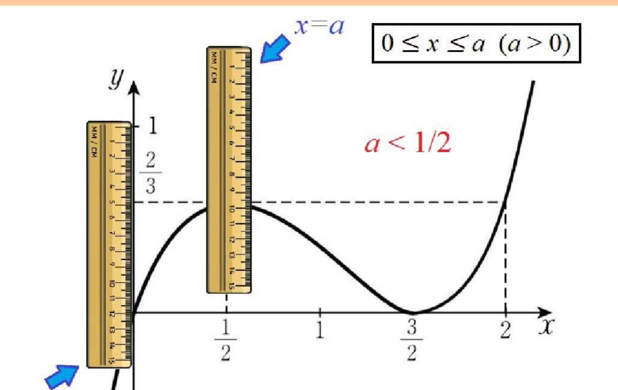
The minimum value is -10, at $x = -2$.

When the **domain** is **arbitrary**:

1) First construct the variation table. Determine the critical points and draw the graph of the function.

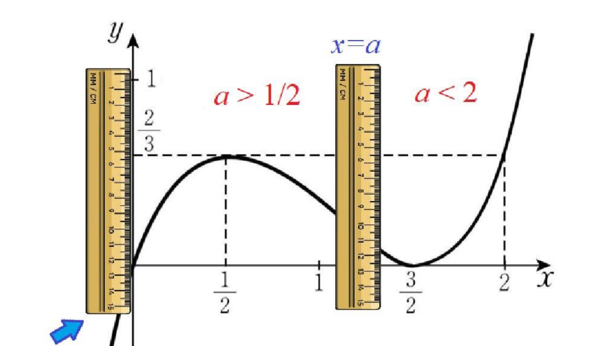
2) Prepare two rulers, pretend that the rulers are the left and right end of the interval. Place both rulers on the graph. Move them <u>from left to right</u>.

3) Observe how the maximum/minimum value changes as the ruler passes through certain values.



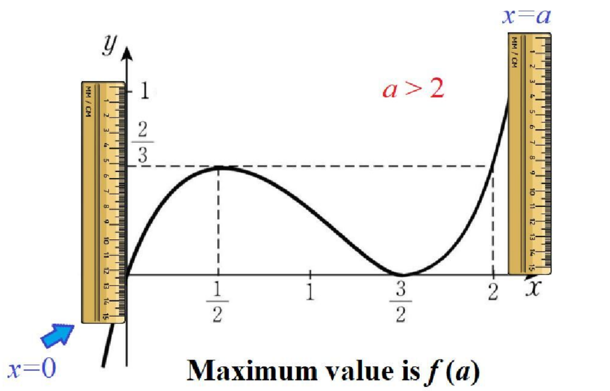
Maximum value is *f* (*a*)

If we shift $x=a$ ruler to the right, the maximum point no longer stays at $x=a$



Maximum value is *f* (1/2)

Move the $x=a$ ruler to the right again, the maximum point switches from $x=1/2$ to $x=a$



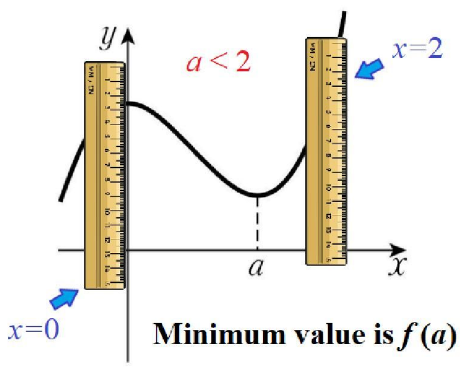
Maximum value is $f(a)$

When the **function** is **arbitrary**:

Similar to above. First draw the graph of the function, indicate the relative maximum and minimum points on the graph. Since it could be *hard* to visualise the curve changing, we may make the intervals change. In this case, we move the rulers from right to left.

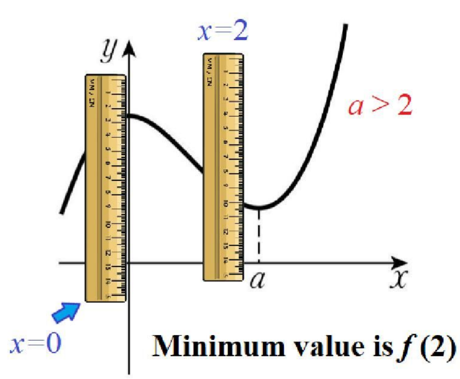
$f(x)=2x^3-3ax^2+2a^3$ on the interval $0 \leqslant x \leqslant 2$

Assume that $a>0$.



Minimum value is $f(a)$

If we move the $x=2$ ruler to the left, the minimum point no longer stays at $x=a$



Minimum value is $f(2)$

Note: Refer to the worksheets for more variations.

L 101-110 : Applications to Equations & Inequalities

To find the number of (real) roots of an equation, we draw the graph of the corresponding function.

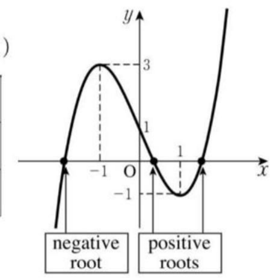
$x^3-3x+1=0$

Let $f(x)=x^3-3x+1$

$f'(x)=3x^2-3=3(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	↗	3	↘	-1	↗

$f(0)=1$



From the graph, there are 2 positive roots and 1 negative root.

When there is an arbitrary constant in the equation, draw the graph of the function and place a ruler on the graph. Move the ruler from bottom to top and observe how the number of solutions changes.

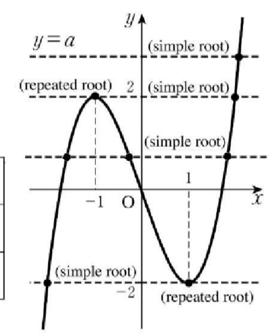
Given the cubic equation $x^3-3x-a=0$

From $x^3-3x=a$,

let $\begin{cases} y=a & \cdots① \\ y=x^3-3x & \cdots② \end{cases}$

From ②, $y'=3(x+1)(x-1)$

x	$\cdots$	-1	$\cdots$	1	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	2	↘	-2	↗



From the graph,

When $a<-2$, there is 1 negative real root.

When $a=\pm 2$, there are 2 different real roots.

When $-2<a<2$, there are 3 different real roots.

When $a>2$, there is 1 positive real root.

It is also convenient to find the number of common points between the curve and the x-axis. (L105, 106)

To prove an inequality, let $f(x)$ = LHS – RHS and show that $f(x)$ lies above the x-axis for the given x.

Prove that when $x>0$, $2x^3+12x+1>9x^2$.

Let $f(x)=2x^3-9x^2+12x+1$

$f'(x)=6x^2-18x+12=6(x-2)(x-1)$

x	0	$\cdots$	1	$\cdots$	2	$\cdots$
$f'(x)$	$+$	$+$	0	$-$	0	$+$
$f(x)$	1	↗	6	↘	5	↗

As $f(0)=1$ and $f(2)=5$, $f(x)>0$ for $x>0$.

Thus, when $x>0$, $2x^3-9x^2+12x+1>0$

L 111-120 : Indefinite & Definite Integrals

A function that, when differentiated, becomes $f(x)$ is known as an **indefinite integral** of $f(x)$.

Formula

When n is a positive integer or zero,

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

(C is the constant of integration.)

Properties of Indefinite Integrals

1. $\int kf(x)dx = k\int f(x)dx$ (k is a constant.)
2. $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
3. $\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$

When the indefinite integral of $f(x)$ is $F(x)$, then $F(b) - F(a)$ is known as the **definite integral** of $f(x)$ from a to b, written as $\int_a^b f(x)\,dx$.

Definition of Definite Integral

When the indefinite integral of $f(x)$ is $F(x)$,

$$\int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Example:

$$\int_1^3 (x+2)^2 dx = \int_1^3 (x^2+4x+4)dx = \left[\frac{1}{3}x^3 + 2x^2 + 4x\right]_1^3$$

$$= (9+18+12) - \left(\frac{1}{3}+2+4\right) = \mathbf{\frac{98}{3}}$$

L 121-140 : Definite Integrals

Properties of Definite Integrals

1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ (k is a constant.)
2. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
3. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
4. $\int_{-a}^a x^n dx = \begin{cases} 0 & (\text{When } n \text{ is } 1, 3, 5, \dots) \\ 2\int_0^a x^n dx & (\text{When } n \text{ is } 0, 2, 4, \dots) \end{cases}$
5. $\int_\alpha^\beta (x-\alpha)(x-\beta)dx = -\frac{1}{6}(\beta-\alpha)^3$
6. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
7. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

Some variations:

In the following example, there is no need to find the roots of the integrand *explicitly*.

Assume that α and β are the two distinct real roots of the integrand, and $\alpha < \beta$.

$$\int_\alpha^\beta (x^2-2x-1)dx$$

$\alpha+\beta = 2$, $\alpha\beta = -1$ root-coefficient relationship

From $(\beta-\alpha)^2 = (\alpha+\beta)^2 - 4\alpha\beta = 8$,

$\beta - \alpha = 2\sqrt{2}$

$$\int_\alpha^\beta (x^2-2x-1)dx = -\frac{1}{6}(\beta-\alpha)^3$$

$$= -\frac{1}{6}(2\sqrt{2})^3 = -\frac{8}{3}\sqrt{2}$$

Recall modulus functions (L31–40)

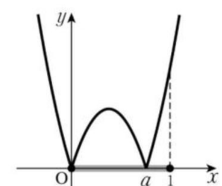
Evaluate the integral $I = \int_0^1 |x(x-a)|dx$ when $0 < a < 1$.

$$|x(x-a)| = \begin{cases} -x(x-a) & (0 \leq x \leq a) \\ x(x-a) & (a \leq x \leq 1) \end{cases}$$

$$I = \int_0^a [-x(x-a)]dx + \int_a^1 x(x-a)dx$$

$$= -\left[\frac{1}{3}x^3 - \frac{a}{2}x^2\right]_0^a + \left[\frac{1}{3}x^3 - \frac{a}{2}x^2\right]_a^1$$

$$= \frac{a^3}{3} - \frac{a}{2} + \frac{1}{3}$$

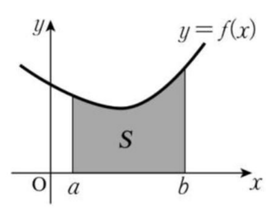


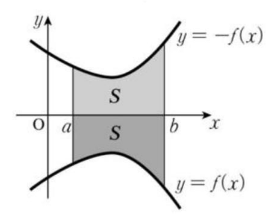
L 141-160 : Areas

Given that the curve $y = f(x)$ intersects the x-axis at 2 points, where the x-coordinates are α and β ($\alpha < \beta$), the area, S, enclosed by the curve and the x-axis can be expressed as follows:

(i) When the region is above the x-axis: $S = \int_\alpha^\beta f(x)dx$

(ii) When the region is below the x-axis: $S = -\int_\alpha^\beta f(x)dx$

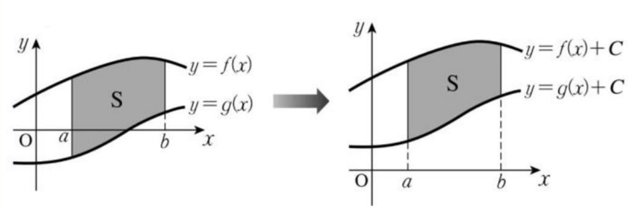




To find the area enclosed by two curves, we can **translate** them and the area will remain unaltered.

The area, enclosed by $y = f(x)$ and $y = g(x)$ is:

$$\boxed{S = \int_a^b [f(x)-g(x)]dx \quad (\text{where } f(x) \geq g(x))}$$



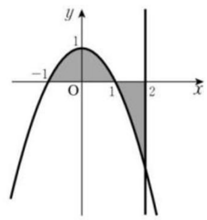
Find the area, S, of the shaded region ( ) enclosed by the given curve, the given line and the x-axis.

$y=-x^2+1,\quad$ line $x=2$

From $y=-x^2+1=-(x+1)(x-1)=0$,

$x=-1,\ 1$

$$S=\int_{-1}^{1}(-x^2+1)dx-\int_{1}^{2}(-x^2+1)dx$$
$$=2\int_{0}^{1}(-x^2+1)dx-\int_{1}^{2}(-x^2+1)dx$$
$$=2\Big[-\frac{1}{3}x^3+x\Big]_0^1-\Big[-\frac{1}{3}x^3+x\Big]_1^2=\mathbf{\frac{8}{3}}$$

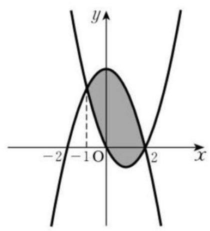


$y=x^2-2x,\qquad y=-x^2+4$

From $x^2-2x=-x^2+4$,

$x=-1,\ 2$

$$S=\int_{-1}^{2}[-x^2+4-(x^2-2x)]dx$$
$$=\int_{-1}^{2}(-2x^2+2x+4)dx$$
$$=\Big[-\frac{2}{3}x^3+x^2+4x\Big]_{-1}^{2}=\mathbf{9}$$



When the region is enclosed by more than 2 curves/ lines, try to partition the region so that each region is enclosed by exactly two curves (and the boundaries parallel to the y-axis).

Find the area, S, enclosed by $y=x^2-3x$, $y=2x$, and $y=-x$.

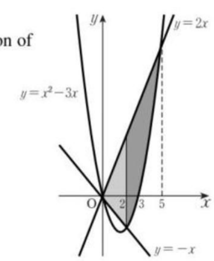
Finding the x-coordinates of the points of intersection of $y=x^2-3x$ and $y=-x$:

$x=0,\ 2$

Finding the x-coordinates of the points of intersection of $y=x^2-3x$ and $y=2x$:

$x=0,\ 5$

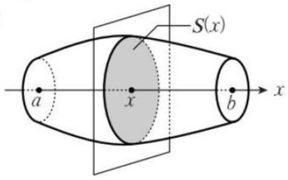
$$S=\underbrace{\int_{0}^{2}[2x-(-x)]dx}_{\text{This represents the area of the light shaded region}}+\underbrace{\int_{2}^{5}[2x-(x^2-3x)]dx}_{\text{This represents the area of the dark shaded region}}$$
$$=\int_{0}^{2}3xdx+\int_{0}^{2}(-x^2+5x)dx$$
$$=\Big[\frac{3}{2}x^2\Big]_0^2+\Big[-\frac{1}{3}x^3+\frac{5}{2}x^2\Big]_2^5=\mathbf{\frac{39}{2}}$$



L 161-170 : Volumes

If $S(x)$ is a function of x representing the cross-sectional area of a given solid, cut by a plane perpendicular to the x-axis, then the volume for $a\leqslant x\leqslant b$ is:

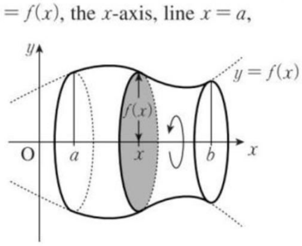
$$\boxed{V=\int_a^b S(x)dx}$$



The solid formed by rotating a line or curve around an axis is called a ***solid of revolution***. Let V be the volume of the solid formed by rotating the region enclosed by the function $y=f(x)$, the x-axis, line $x=a$, and line $x=b$.

The volume, V, of the solid of revolution for $a\leqslant x\leqslant b$ is:

$$\boxed{V=\int_a^b \pi y^2dx=\pi\int_a^b[f(x)]^2dx}$$



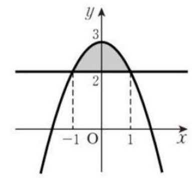
Examples:

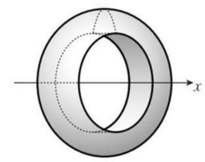
Find the volume, V, of the solid formed by rotating the region enclosed by $y=3-x^2$ and the line $y=2$ about the x-axis.

Finding the x-coordinates of the points of intersection between $y=3-x^2$ and $y=2$,

$x=\pm1$

$$V=\pi\int_{-1}^{1}(3-x^2)^2dx-\pi\int_{-1}^{1}2^2dx$$
$$=\pi\int_{-1}^{1}(x^4-6x^2+5)dx$$
$$=2\pi\int_{0}^{1}(x^4-6x^2+5)dx$$
$$=2\pi\Big[\frac{1}{5}x^5-2x^3+5x\Big]_0^1$$
$$=\mathbf{\frac{32}{5}\pi}$$





L 171-180 : Velocity & Distance

Velocity is the instantaneous rate of change of position with respect to time

If the position of point P is $x=f(t)$, we can express the velocity of point P as:

$$\boxed{v=\frac{dx}{dt}=f'(t)}$$

Applying the fundamental theorem of calculus to the above equation, we have

The displacement (the change in position) of point P from $t=a$ to $t=b$ can be found as follows:

$$\boxed{f(b)-f(a)=\int_a^b f'(t)dt=\int_a^b vdt}$$

Distance travelled is the sum of absolute value of all displacements over a time interval.

Given a point P moving at time t with velocity v, the distance, s, travelled from time $t=a$ to $t=b$ is expressed as:

$$\boxed{s=\int_a^b|v|dt}$$

Note: Distance travelled is always positive (or zero).
Displacements may be positive or negative (or zero).

Note: Refer to the worksheets for more variations.

L 181-200 : Summary of Differentiation, Integration

This particular set includes **challenging problems** that require higher order thinking skills. Make sure to go through the problems (and solutions) *carefully* with your instructor.

If you are facing difficulties, you may drop me a message on Pinterest (Peter Chang).